

**SYMPOSIUM**

**OCTOBER**



**ON THE USE OF**

**1-5 1962**

**computers in civil engineering**

**Laboratório Nacional de Engenharia Civil**

**Lisbon - Portugal**

## Conclusion –

January 2014

As Ed reviews his last 60 years of professional activity, it is apparent that the symposium in Lisbon, Portugal was one of the most significant events in his life.

First; Ray Clough and Ed presented the results of the analysis of a large dam which clearly illustrated the superiority of the finite element method compared to any other numerical or experimental approach used in world at that time. This paper was included in the proceedings of the conference which would have very limited distribution. However, the editor of the international Bulletin RILEM (*R*eunion *I*nternationale des *L*aboratoires et *E*xperts des *M*ateriaux was formed in 1948 and is still functioning) asked Ray and the symposium chairmen if RILEM could publish the paper in their next edition of the bulletin. See the link shown below to read the paper and to see the old FORTRAN program that automated finite element analyses:

<http://edwilson.org/History/The%20First%20Automated%20Finite%20Element%20Program.pdf>

Second; Ed presented his first research paper in the field of Earthquake Engineering. During the previous three years Ray and Ed (working as consultants for several structural engineering firms in San Francisco) conducted linear seismic time-history analysis of different types of multistory building. They used an early version of SMIS (*S*ymbolic *M*atrix *I*nterpretive *S*ystem written by Ed to teach static and dynamic analysis of structures using matrix notation) to create the stiffness matrix. The dynamic analysis was then reduced to a series of standard matrix operations using the mode superposition method. However, they realize that in order to perform nonlinear dynamic analysis of structures they would need to use a step-by-step integration method.

At that time most dynamic linear and nonlinear structural analysis was conducted using Newmark's linear acceleration method in which the equations were solved by iteration at each time step. Therefore, Ray suggested Ed use the Newmark's method and compare results with the mode superposition method for linear structures. This paper can be read shown at

The major contribution of this paper was to prove iteration was not required for dynamic analysis by the step-by-step approach. In addition, they demonstrated the two methods produced almost identical results if the same damping matrix was used by each method. Check the following link to read Ed's recent recommendations on the nonlinear dynamic analysis of structures:

<http://edwilson.org/BOOK-Wilson/18-FMA.pdf>

# DYNAMIC RESPONSE BY STEP-BY-STEP MATRIX ANALYSIS

by

Edward L. Wilson<sup>1</sup> and Ray W. Clough<sup>2</sup>

## SUMMARY

The step-by-step integration procedure provides one of the most powerful digital computer techniques available for calculating the response of lumped-mass systems to arbitrary dynamic loads. In this paper, the step-by-step analysis is formulated as a sequence of matrix operations. A unique feature of the development permits the treatment of accelerations which vary linearly during each time interval without need for the iterative operations required by other procedures.

Use of the method is demonstrated with a practical example. Also presented is a comparison with the mode-superposition method with regard to relative computer storage requirements, accuracy and time required for computation.

## INTRODUCTION

Although the theory of structural dynamics has long been understood, the problem of evaluating the response of complex structures to arbitrary dynamic loads was generally considered intractable from a computational standpoint until electronic digital computers were applied to the task. With the aid of digital computers, however, almost any class of dynamic response problem may now be undertaken.

In most practical cases the structure which is analyzed is a lumped parameter idealization of an actual structure, in which the mass properties of the system are assumed to be separated from its elastic characteristics for the purpose of the mathematical formulation. Typically, this idealization involves the assumption that the mass is concentrated at discrete points such that the displacements of these

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masses may be described by a limited number of displacement coordinates. The advantage of this lumped-mass idealization is that the equilibrium conditions imposed on the system may be expressed by a finite number of ordinary differential equations rather than the partial differential equation required to describe the actual distributed mass system.

These simultaneous differential equations representing the equilibrium of the lumped-mass system may be expressed most conveniently as a matrix equation. Then either of the two entirely different approaches to the solution of this equation may be adopted. The mode-superposition method<sup>(1)\*</sup> involves the solution of the characteristic value problem represented by the free vibration response of the system, followed by the transformation to the principal coordinates determined as the characteristic shapes of the system. This procedure uncouples the response of the system, so that the response of each coordinate may be evaluated independently of the others. The second method of dynamic analysis is called the step-by-step method, and involves the direct numerical integration of the equilibrium equations in their original form, without transformation to the principal coordinates of the system.

One of the principal advantages of the mode-superposition method lies in the fact that the response of the system is largely expressed by the first few modes of vibration; thus, good accuracy may be obtained by this method from an analysis involving only a few of the principal coordinates, while all coordinates must be retained in the step-by-step method. On the other hand, the evaluation of the characteristic value problem and transformation to the principal coordinates are major computational problems not required of the step-by-step method. Furthermore, the mode-superposition method is based on the assumption of linear structural behavior, whereas the step-by-step method may be applied to non-linear systems simply by modifying the assumed linear properties approximately at each successive step of integration. Thus, it is clear that there is need for both methods of dynamic analysis.

The purpose of this paper is to describe a matrix formulation of the step-by-step integration procedure. In this form, the method is applicable to the analysis of systems having any number of degrees of freedom. Furthermore, from a computational standpoint it has the advantage that the dynamic analysis program may be incorporated into a general matrix interpretive scheme, and thus it becomes merely another operation which may be applied in the matrix analysis of a

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\* Numbers in parentheses refers to similarly numbered References listed at the end of the paper.

structure. Step-by-step procedures are not new, of course, and have been investigated intensively by other authors. (2,3) However, it is believed that the formulation of the method presented here, which permits the treatment of linear (or higher order) variations of accelerations during the time increment without iteration, has definite advantages over techniques which have been described before.

## DERIVATION OF STEP-BY-STEP FORMULAS

### Equilibrium Equation

Assuming a viscous form of damping, the equilibrium of a discrete mass system at time "t" is expressed by the following matrix equation:

$$[M]\{\ddot{X}\}_t + [C]\{\dot{X}\}_t + [K]\{X\}_t = \{P\}_t \quad (1)$$

where

- $\{X\}_t$  = the displacement of the system
- $\{\dot{X}\}_t$  = the velocity of the system
- $\{\ddot{X}\}_t$  = the acceleration of the system
- $\{P\}_t$  = the force acting on the system
- $[M]$  = the mass matrix
- $[C]$  = the damping matrix
- $[K]$  = the stiffness matrix

The solution of this set of second order differential equations may be reduced to a sequence of recurring matrix equations if an assumption is made as to the behavior of the system within a small increment of time.

### Linear Acceleration Method

It will be assumed first that the acceleration associated with each degree of freedom of the discrete mass system varies linearly within a time increment,  $\Delta t$ . This assumption, which is illustrated in Fig. 1, leads to a parabolic variation of velocity and a cubic variation of displacement within the time increment.

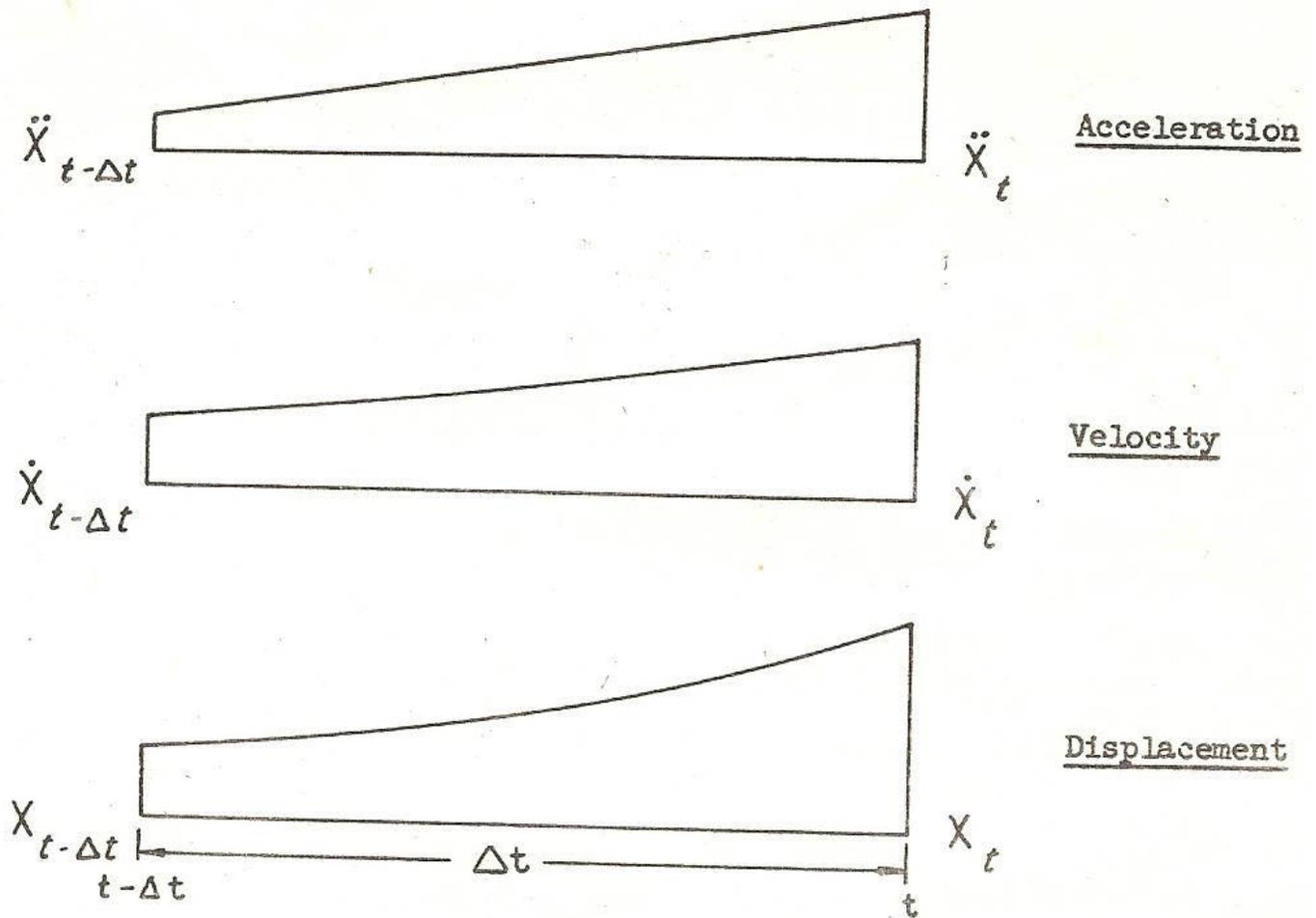


Fig. 1 - Assumed Behavior of Typical Displacement Component

From a direct integration over the time interval for all mass points, the following matrix equations for velocity and displacement at the end of the time interval are obtained:

$$\{\dot{X}\}_t = \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t}{2} \{\ddot{X}\}_{t-\Delta t} + \frac{\Delta t}{2} \{\ddot{X}\}_t \quad (2a)$$

$$\{X\}_t = \{X\}_{t-\Delta t} + \Delta t \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{3} \{\ddot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{6} \{\ddot{X}\}_t \quad (2b)$$

The substitution of Eq. 2a and 2b into Eq.1 yields the following equation for the acceleration at the end of the time interval:

$$\{\ddot{X}\}_t = [F] \left\{ \{P\}_t - [C] \{a\} - [K] \{b\} \right\} \quad (3)$$

$$[F] = \left[ [M] + \frac{\Delta t}{2}[C] + \frac{\Delta t^2}{6}[K] \right]^{-1}$$

$$\{a\} = \{\ddot{X}\}_{t-\Delta t} + \frac{\Delta t}{2} \{\ddot{X}\}_{t-\Delta t}$$

$$\{b\} = \{X\}_{t-\Delta t} + \Delta t \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{3} \{\ddot{X}\}_{t-\Delta t}$$

From Eq. 2a the velocity at the end of the interval is given by

$$\{\dot{X}\}_t = \{a\} + \frac{\Delta t}{2} \{\ddot{X}\}_t \quad (4)$$

From Eq. 2b the displacement at the end of the interval is given by

$$\{X\} = \{b\} + \frac{\Delta t^2}{6} \{\ddot{X}\}_t \quad (5)$$

It is important to note that for linear problems the matrix  $[F]$  need only be formed once since it is independent of time.

The solution of Eq. 1 proceeds as follows. The initial displacement  $\{X\}_0$  and the initial velocity  $\{\dot{X}\}_0$  are given as the initial conditions of the problem. The initial acceleration  $\{\ddot{X}\}_0$  is obtained from Eq. 1 as

$$\{\ddot{X}\}_0 = [M]^{-1} \{ \{P\}_0 - [C]\{\dot{X}\}_0 - [K]\{X\}_0 \} \quad (6)$$

Then the step-by-step response of the system is given by repeated application of Eqs. 3, 4 and 5.

Table 1 summarizes the essential equations of the linear acceleration method.

TABLE 1 - Summary of Linear Acceleration Method

INITIAL CALCULATIONS :

$$\{\ddot{X}\}_0 = [M]^{-1} \{ \{P\}_0 - [C]\{\dot{X}\}_0 - [K]\{X\}_0 \}$$

$$[F] = \left[ [M] + \frac{\Delta t}{2} [C] + \frac{\Delta t^2}{6} [K] \right]^{-1} = [\bar{K}]^{-1}$$

STEP-BY-STEP EQUATIONS:

$$\{\ddot{X}\}_t = [F] \{ \{P\}_t - [C]\{a\} - [K]\{b\} \}$$

$$\{\dot{X}\}_t = \{a\} + \frac{\Delta t}{2} \{\ddot{X}\}_t$$

$$\{X\}_t = \{b\} + \frac{\Delta t^2}{6} \{\ddot{X}\}_t$$

WHERE

$$\{a\} = \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t}{2} \{\ddot{X}\}_{t-\Delta t}$$

$$\{b\} = \{X\}_{t-\Delta t} + \Delta t \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{3} \{\ddot{X}\}_{t-\Delta t}$$

(3) Parabolic Acceleration Method

It is possible, by a procedure similar to the one presented above, to develop a set of formulas based on a parabolic variation of acceleration within a small increment of time. The derivation of these equations is not given here. Only the results are summarized in Table II. This method requires an increase in the number of operations per cycle; however, as a result of the increase in accuracy the time increment may be increased. The use of this higher order method must be justified by the characteristics of the particular problem to be solved.

TABLE II - Summary of Parabolic Acceleration Method

INITIAL CALCULATIONS:

$$[F] = \left[ [M] + \frac{\Delta t}{3} [C] + \frac{\Delta t^2}{12} [K] \right]^{-1}$$

$$\{\ddot{X}\}_0 = [M]^{-1} \left\{ \{P\}_0 - [C] \{\dot{X}\}_0 - [K] \{X\}_0 \right\}$$

$$\{\ddot{\dot{X}}\}_0 = [M]^{-1} \left\{ \{\dot{P}\}_0 - [C] \{\ddot{X}\}_0 - [K] \{\dot{X}\}_0 \right\}$$

WHERE

$$\{\dot{P}\}_0 = \frac{1}{\Delta t} \left\{ \{P\}_{\Delta t} - \{P\}_0 \right\}$$

STEP-BY-STEP EQUATIONS:

$$\{\ddot{X}\}_t = [F] \left\{ \{P\}_t - [C] \{a\} - [K] \{b\} \right\}$$

$$\{\dot{X}\}_t = \{a\} + \frac{\Delta t}{3} \{\ddot{X}\}_t$$

$$\{X\}_t = \{b\} + \frac{\Delta t^2}{12} \{\ddot{X}\}_t$$

$$\{\ddot{\dot{X}}\}_t = \frac{2}{\Delta t} \{\ddot{X}\}_t - \frac{2}{\Delta t} \{\ddot{X}\}_{t-\Delta t} - \{\ddot{\dot{X}}\}_{t-\Delta t}$$

WHERE

$$\{a\} = \{\dot{X}\}_{t-\Delta t} + \frac{2}{3} \Delta t \{\ddot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{6} \{\ddot{\dot{X}}\}_{t-\Delta t}$$

$$\{b\} = \{X\}_{t-\Delta t} + \Delta t \{\dot{X}\}_{t-\Delta t} + \frac{5}{12} \Delta t^2 \{\ddot{X}\}_{t-\Delta t} + \frac{\Delta t^3}{12} \{\ddot{\dot{X}}\}_{t-\Delta t}$$

### Simplification of [C] Matrix

For most practical structures the exact form of damping is unknown. Since its effect on the response of a structure is generally small, a simplifying assumption as to its form is justifiable. It has been suggested that the [C] matrix can be replaced by the following matrix relationship<sup>(4)</sup>:

$$[C] = \alpha [M] + \beta [K] \quad (7)$$

The substitution of Eq. 7 into the set of equations presented in Table I yields a modified set of equations which is shown in Table III.

TABLE III - Summary of Linear Acceleration Method With Simplified Form of Damping

INITIAL CALCULATIONS:

$$\{\ddot{X}\}_0 = [M]^{-1} \left\{ \{P\}_0 - \alpha [M] \{\dot{X}\}_0 - [K] \left\{ \{X\}_0 + \beta \{\dot{X}\}_0 \right\} \right\}$$

$$[F] = \left[ \left( 1 + \alpha \frac{\Delta t}{2} \right) [M] + \left( \frac{\Delta t^2}{6} + \beta \frac{\Delta t}{2} \right) [K] \right]^{-1}$$

STEP-BY-STEP EQUATIONS:

$$\{\ddot{X}\}_t = [F] \left\{ \{P\}_t - \alpha [M] \{a\} - [K] \left\{ \{b\} + \beta \{a\} \right\} \right\}$$

$$\{\dot{X}\}_t = \{a\} + \frac{\Delta t}{2} \{\ddot{X}\}_t$$

$$\{X\}_t = \{b\} + \frac{\Delta t^2}{2} \{\ddot{X}\}_t$$

WHERE

$$\{a\} = \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t}{2} \{\ddot{X}\}_{t-\Delta t}$$

$$\{b\} = \{X\}_{t-\Delta t} + \Delta t \{\dot{X}\}_{t-\Delta t} + \frac{\Delta t^2}{3} \{\ddot{X}\}_{t-\Delta t}$$

#### Simplification of Load Matrix

In general, the applied loads which are associated with each degree of freedom may each be an independent function of time. However, for most practical problems this generality is not necessary and the load at any time  $t$  may be placed in the following form:

$$\{p\}_t = S_t \{P\} \quad (8)$$

where  $\{P\}$  prescribes the distribution of load on the structure and  $S_t$  is a scalar multiplier indicating the load amplitude at time  $t$ .

In the case of an earthquake loading which is given in terms of the ground acceleration, it can be shown easily that the matrix  $\{P\}$  is simply a vector of the masses which are associated with the lateral components of displacements and that  $S_t$  is the ground acceleration at time  $t$ .

### Element Forces

After the determination of the displacement  $\{X\}_t$  the forces  $\{F\}_t$  in the individual structural elements may be obtained from the matrix equation:

$$\{F\}_t = [T]\{X\}_t \quad (9)$$

where  $[T]$  is the same element force-displacement transformation matrix that would be used in the case of static loading. The form of this matrix is not given here since it depends on the specific structure that is being analyzed, but it may be derived by standard methods of matrix structural analysis<sup>(5)</sup>.

### Selection of Time Increment

The selection of the time increment  $\Delta t$  must be based on a consideration of the properties of the structure as well as the form of the applied load. For the linear acceleration method, Newmark<sup>(2)</sup> has suggested that  $\Delta t$  should be less than 1/10 of the smallest period of the structure. Since the smallest period of the structure is not always known, perhaps the simplest procedure is to try several different time increments until the desired degree of accuracy is obtained. It is apparent that this approach is extremely inefficient; however, this disadvantage affects only the length of computation, which on modern digital computers is normally small compared with the time required for complete formulation of the problem.

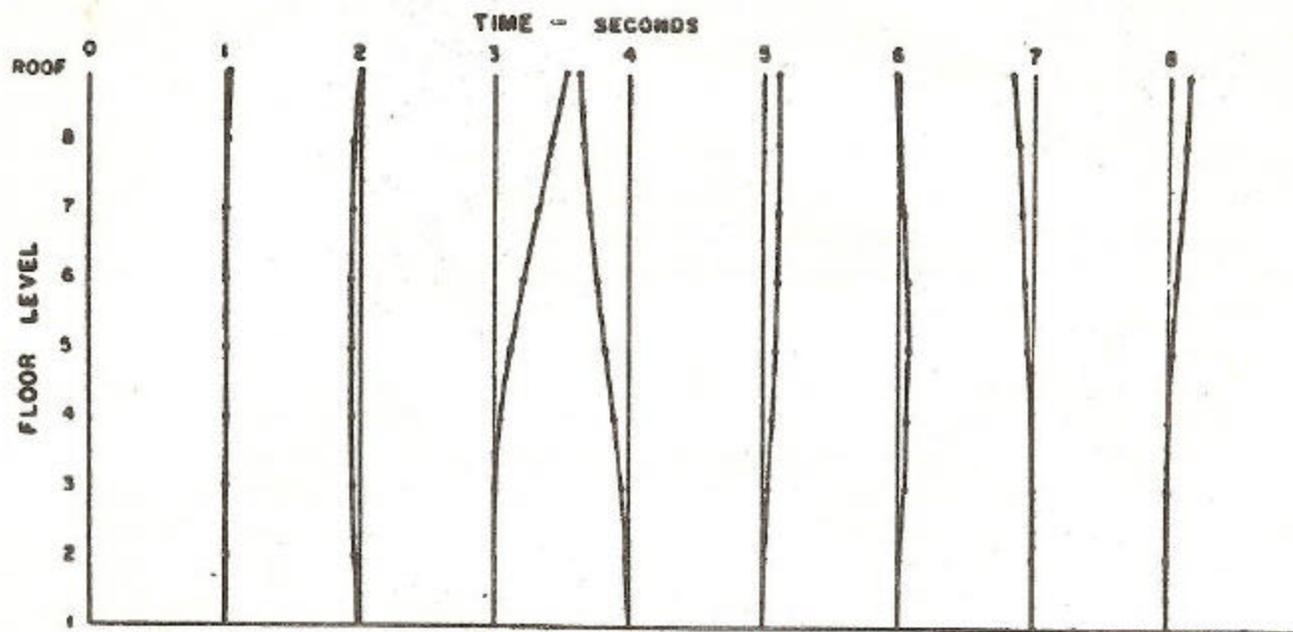
## COMPUTER PROGRAM

Since the step-by-step methods presented here involve only standard matrix operations they are easily programmed for the digital computer. The generality of the program must depend on its intended use as well as the type of computer used.

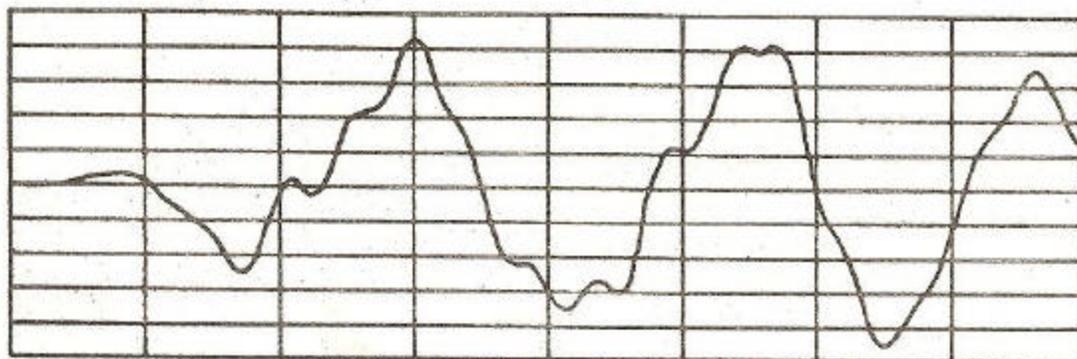
### Computer Storage

At any step in the analysis the matrices that must be retained in

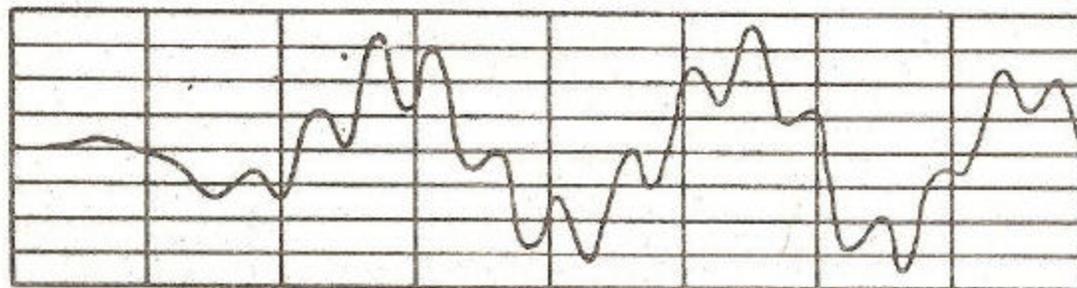




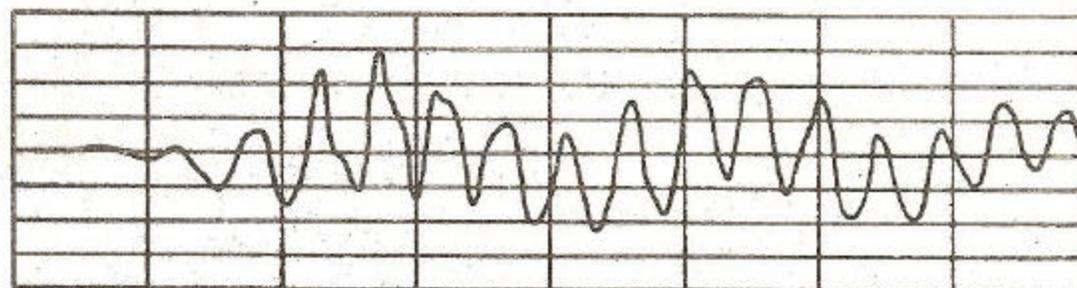
(a) DEFLECTED SHAPE



(b) DISPLACEMENT OF ROOF



(c) BASE MOMENT



(d) BASE SHEAR

FIG. 2 RESPONSE OF BUILDING SUBJECTED TO EARTHQUAKE LOADING

The mass of the structure is lumped at eight levels and is given by

$$[M] = \begin{bmatrix} 2.76 & 2.76 & 2.76 & 2.73 & 2.69 & 2.69 & 2.69 & 2.63 \end{bmatrix}$$

The ground acceleration of an earthquake was used as the loading for the structure. The damping coefficients  $\alpha$  and  $\beta$  were set equal to 0.50/sec. and 0.001 sec. The time increment used was 0.0125 sec.

The displacements, moments, and shears for the first eight seconds of the response were determined at each story; however, only a few of these response data are shown in Fig. 2.

As a check on the step-by-step method, the same structure was solved using the mode-superposition method. The solution of the characteristic value problem yields the following values for the frequencies of the structure:

$$[\omega_i] = \begin{bmatrix} 2.894 & 13.65 & 30.79 & 48.66 & 65.17 & 78.78 & 89.03 & 95.22 \end{bmatrix} \frac{\text{rad.}}{\text{sec.}}$$

The damping associated with each mode is determined from the following equation: (4)

$$\lambda_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \quad (10)$$

The results obtained by the mode-superposition method and the step-by-step method differed at any time by less than two percent. Within the accuracy of plotting, Fig. 2 illustrates the results for both methods.

## CONCLUSIONS

The step-by-step method as presented in this paper provides a systematic procedure to determine the dynamic response of discrete mass systems subjected to arbitrary loading. The formulation of the step-by-step equations in matrix form allows the response of a system to be determined by a repeated sequence of matrix operations. Furthermore, the iteration on each time increment, which is required by other procedures, is eliminated by an initial matrix inversion.

#### REFERENCES

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