1.

MATERIAL PROPERTIES

Material Properties Must Be Evaluated
By Laboratory or Field Tests

1.1 INTRODUCTION

The fundamental equations of structural mechanics can be placed in three categories[1]. First, the stress-strain relationship contains the material property information that must be evaluated by laboratory or field experiments. Second, the total structure, each element, and each infinitesimal particle within each element must be in force equilibrium in their deformed position. Third, displacement compatibility conditions must be satisfied.

If all three equations are satisfied at all points in time, other conditions will automatically be satisfied. For example, at any point in time the total work done by the external loads must equal the kinetic and strain energy stored within the structural system plus any energy that has been dissipated by the system. Virtual work and variational principles are of significant value in the mathematical derivation of certain equations; however, they are not fundamental equations of mechanics.

1.2 ANISOTROPIC MATERIALS

The linear stress-strain relationships contain the material property constants, which can only be evaluated by laboratory or field experiments. The mechanical material properties for most common material, such as steel, are well known and are defined in terms of three numbers: modulus of elasticity E, Poisson's ratio

ν and coefficient of thermal expansion α. In addition, the unit weight w and the unit mass ρ are considered to be fundamental material properties.

Before the development of the finite element method, most analytical solutions in solid mechanics were restricted to materials that were isotropic (equal properties in all directions) and homogeneous (same properties at all points in the solid). Since the introduction of the finite element method, this limitation no longer exists. Hence, it is reasonable to start with a definition of anisotropic materials, which may be different in every element in a structure.

The positive definition of stresses, in reference to an orthogonal 1-2-3 system, is shown in Figure 1.1.

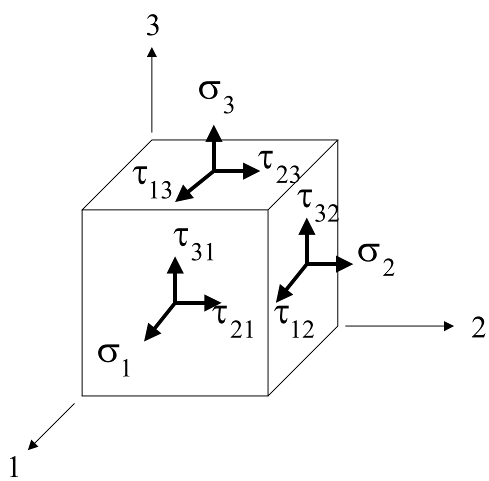

Figure 1.1 Definition of Positive Stresses

All stresses are by definition in units of force-per-unit-area. In matrix notation, the six independent stresses can be defined by:

$$\mathbf{f}^{\mathbf{T}} = [\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \tau_{21} \quad \tau_{31} \quad \tau_{23}] \tag{1.1}$$

From equilibrium, $\tau_{12} = \tau_{21}$, $\tau_{31} = \tau_{13}$ and $\tau_{32} = \tau_{23}$. The six corresponding engineering strains are:

$$\mathbf{d}^{\mathbf{T}} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \gamma_{21} & \gamma_{31} & \gamma_{23} \end{bmatrix} \tag{1.2}$$

The most general form of the three dimensional strain-stress relationship for linear structural materials subjected to both mechanical stresses and temperature change can be written in the following matrix form[2]:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{21} \\ \gamma_{31} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & -\frac{\nu_{14}}{E_4} & -\frac{\nu_{15}}{E_5} & -\frac{\nu_{16}}{E_6} \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & -\frac{\nu_{24}}{E_4} & -\frac{\nu_{25}}{E_5} & -\frac{\nu_{26}}{E_6} \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & -\frac{\nu_{34}}{E_4} & -\frac{\nu_{35}}{E_4} & -\frac{\nu_{36}}{E_6} \\ -\frac{\nu_{41}}{E_1} & -\frac{\nu_{42}}{E_2} & -\frac{\nu_{43}}{E_3} & \frac{1}{E_4} & -\frac{\nu_{45}}{E_5} & -\frac{\nu_{46}}{E_6} \\ -\frac{\nu_{51}}{E_1} & -\frac{\nu_{52}}{E_2} & -\frac{\nu_{53}}{E_3} & -\frac{\nu_{54}}{E_4} & \frac{1}{E_5} & -\frac{\nu_{56}}{E_6} \\ -\frac{\nu_{61}}{E_1} & -\frac{\nu_{62}}{E_2} & -\frac{\nu_{63}}{E_3} & -\frac{\nu_{64}}{E_4} & -\frac{\nu_{65}}{E_5} & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{21} \\ \tau_{31} \\ \tau_{23} \end{bmatrix} + \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_{21} \\ \alpha_{31} \\ \alpha_{23} \end{bmatrix} \tag{1.3}$$

Or, in symbolic matrix form:

$$\mathbf{d} = \mathbf{Cf} + \Delta T\mathbf{a} \tag{1.4}$$

The $\mathbf{C}$ matrix is known as the compliance matrix and can be considered to be the most fundamental definition of the material properties because all terms can be evaluated directly from simple laboratory experiments. Each column of the $\mathbf{C}$ matrix represents the strains caused by the application of a unit stress. The temperature increase ΔT is in reference to the temperature at zero stress. The $\mathbf{a}$ matrix indicates the strains caused by a unit temperature increase.

Basic energy principles require that the $\mathbf{C}$ matrix for linear material be symmetrical. Hence,

$$\frac{\nu_{ij}}{E_j} = \frac{\nu_{ji}}{E_i} \tag{1.5}$$

However, because of experimental error or small nonlinear behavior of the material, this condition is not identically satisfied for most materials. Therefore, these experimental values are normally averaged so that symmetrical values can be used in the analyses.

1.3 USE OF MATERIAL PROPERTIES WITHIN COMPUTER PROGRAMS

Most of the modern computer programs for finite element analysis require that the stresses be expressed in terms of the strains and temperature change. Therefore, an equation of the following form is required within the program:

$$\mathbf{f} = \mathbf{E}\mathbf{d} + \mathbf{f_0} \tag{1.6}$$

in which $\mathbf{E} = \mathbf{C}^{-1}$. Therefore, the zero-strain thermal stresses are defined by:

$$\mathbf{f_0} = -\Delta T\ \mathbf{E}\mathbf{a} \tag{1.7}$$

The numerical inversion of the 6 x 6 **C** matrix for complex anisotropic materials is performed within the computer program. Therefore, it is not necessary to calculate the **E** matrix in analytical form as indicated in many classical books on solid mechanics. In addition, the initial thermal stresses are numerically evaluated within the computer program. Consequently, for the most general anisotropic material, the basic computer input data will be twenty-one elastic constants, plus six coefficients of thermal expansion.

Initial stresses, in addition to thermal stresses, may exist for many different types of structural systems. These initial stresses may be the result of the fabrication or construction history of the structure. If these initial stresses are known, they may be added directly to Equation (1.7).

1.4 ORTHOTROPIC MATERIALS

The most common type of anisotropic material is one in which shear stresses, acting in all three reference planes, cause no normal strains. For this special case, the material is defined as orthotropic and Equation (1.3) can be written as:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{21} \\ \gamma_{31} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{21} \\ \tau_{31} \\ \tau_{23} \end{bmatrix} + \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.8}$$

For orthotropic material, the **C** matrix has nine independent material constants, and there are three independent coefficients of thermal expansion. This type of material property is very common. For example, rocks, concrete, wood and many fiber reinforced materials exhibit orthotropic behavior. It should be pointed out, however, that laboratory tests indicate that Equation (1.8) is only an approximation to the behavior of real materials.

1.5 ISOTROPIC MATERIALS

An isotropic material has equal properties in all directions and is the most commonly used approximation to predict the behavior of linear elastic materials. For isotropic materials, Equation (1.3) is of the following form:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{21} \\ \gamma_{31} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{21} \\ \tau_{31} \\ \tau_{23} \end{bmatrix} + \alpha\ \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.9}$$

It appears that the compliance matrix has three independent material constants. It can easily be shown that the application of a pure shear stress should result in pure tension and compression strains on the element if it is rotated 45 degrees. Using this restriction, it can be shown that:

$$G = \frac{E}{2(1+\nu)} \tag{1.10}$$

Therefore, for isotropic materials only Young's modulus E and Poisson's ratio ν need to be defined. Most computer programs use Equation (1.10) to calculate the shear modulus if it is not specified.

1.6 PLANE STRAIN ISOTROPIC MATERIALS

If $\varepsilon_1, \gamma_{13}, \gamma_{23}, \tau_{13}, \text{and}\, \tau_{23}$ are zero, the structure is in a state of plane strain. For this case the compliance matrix is reduced to a 3 x 3 array. The cross-sections of many dams, tunnels, and solids with a near infinite dimension along the 3-axis can be considered in a state of plane strain for constant loading in the 1-2 plane. For plane strain and isotropic materials, the stress-strain relationship is:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \overline{E} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} - \alpha \, \Delta T \, \overline{E} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{1.11}$$

where

$$\overline{E} = \frac{E}{(1+\nu)(1-2\nu)} \tag{1.12}$$

For the case of plane strain, the displacement and strain in the 3-direction are zero. However, from Equation (1.8) the normal stress in the 3-direction is:

$$\sigma_3 = \nu\,(\sigma_1 + \sigma_2) - E\alpha\,\Delta T \tag{1.13}$$

It is important to note that as Poisson's ratio, ν , approaches 0.5, some terms in the stress-strain relationship approach infinity. These real properties exist for a nearly incompressible material with a relatively low shear modulus.

1.7 PLANE STRESS ISOTROPIC MATERIALS

If $\sigma_3, \tau_{13}, \text{and} \, \tau_{23}$ are zero, the structure is in a state of plane stress. For this case the stress-strain matrix is reduced to a 3 x 3 array. The membrane behavior of thin plates and shear wall structures can be considered in a state of plane strain for constant loading in the 1-2 plane. For plane stress and isotropic materials, the stress-strain relationship is:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \overline{E} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} - \alpha \, \Delta T \, \overline{E} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{1.14}$$

where

$$\overline{E} = \frac{E}{(1-\nu^2)} \tag{1.15}$$

1.8 PROPERTIES OF FLUID-LIKE MATERIALS

Many different isotropic materials, which have a very low shear modulus compared to their bulk modulus, have fluid-like behavior. These materials are often referred to as nearly incompressible solids. The incompressible terminology is very misleading because the compressibility, or bulk modulus, of these materials is normally lower than other solids. The pressure-volume relationship for a solid or a fluid can be written as:

$$\sigma = \lambda\,\varepsilon \tag{1.16}$$

where λ is the bulk modulus of the material, which must be evaluated by pressure-volume laboratory tests. The volume change ε is equal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, and the hydrostatic pressure σ indicates equal stress in all directions. From Equation (1.9) the bulk modulus can be written in terms of Young's modulus and Poisson's ratio as:

$$\lambda = \frac{E}{3(1-2\nu)} \tag{1.17}$$

For fluids, the bulk modulus is an independent constant, Poisson's ratio is 0.5, and Young's modulus and the shear modulus are zero. For isotropic materials, the bulk modulus and shear modulus are known as Lame's elastic constants and are considered to be fundamental material properties for both solids and fluids. From Equation (1.10), Poisson's ratio and Young's modulus can be calculated from:

$$\nu = \frac{3 - 2\frac{G}{\lambda}}{6 + 2\frac{G}{\lambda}} \quad \text{and} \quad E = 2(1+\nu)G \tag{1.18a and 1.18b}$$

If the shear modulus becomes small compared to the bulk modulus, $\nu \approx 0.5$ and $E \approx 3G$. Table 1.1 summarizes approximate material properties for several common materials.

Table 1.1 Approximate Mechanical Properties of Typical Materials

Material	E Young's Modulus ksi	ν Poisson's Ratio	G Shear Modulus ksi	λ Bulk Modulus ksi	α Thermal Expansion $\times 10^{-6}$	w Weight Density lb/in^3
Steel	29,000	0.30	11,154	16,730	6.5	0.283
Aluminum	10,000	0.33	3,750	7,300	13.0	0.100
Concrete	4,000	0.20	1,667	1,100	6.0	0.087
Mercury	0	0.50	0	3,300	-	0.540
Water	0	0.50	0	300	-	0.036
Water*	0.9	0.4995	0.3	300	-	0.036

* These are approximate properties that can be used to model water as a solid material.

It is apparent that the major difference between liquids and solids is that liquids have a very small shear modulus compared to the bulk modulus, and ***liquids are not incompressible.***

1.9 SHEAR AND COMPRESSION WAVE VELOCITIES

The measurement of compression and shear wave velocities of the material using laboratory or field experiments is another simple method that is often used to define material properties. The compressive wave velocity, V_c, and the shear wave velocity, V_s, are given by:

$$V_c = \sqrt{\frac{\lambda + 2G}{\rho}} \tag{1.19}$$

$$V_s = \sqrt{\frac{G}{\rho}} \tag{1.20}$$

where ρ is the mass density of the material. Therefore, it is possible to calculate all of the other elastic properties for isotropic materials from these equations. It is apparent that shear waves cannot propagate in fluids since the shear modulus is zero.

1.10 AXISYMMETRIC MATERIAL PROPERTIES

A large number of very common types of structures, such as pipes, pressure vessels, fluid storage tanks, rockets, and other space structures, are included in the category of axisymmetric structures. Many axisymmetric structures have anisotropic materials. For the case of axisymmetric solids subjected to non-axisymmetric loads, the compliance matrix, as defined by Equation (1.3), can be rewritten in terms of the r, z and θ reference system as Equation (1.21). The solution of this special case of a three-dimensional solid can be accomplished by expressing the node point displacements and loads in a series of harmonic functions. The solution is then expressed as a summation of the results of a series of two-dimensional, axisymmetric problems[3].

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{z\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & -\frac{\nu_{14}}{E_4} & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & -\frac{\nu_{24}}{E_4} & 0 & 0 \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & -\frac{\nu_{34}}{E_4} & 0 & 0 \\ -\frac{\nu_{41}}{E_1} & -\frac{\nu_{42}}{E_2} & -\frac{\nu_{43}}{E_3} & \frac{1}{E_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{E_5} & -\frac{\nu_{56}}{E_6} \\ 0 & 0 & 0 & 0 & -\frac{\nu_{65}}{E_5} & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \\ \tau_{r\theta} \\ \tau_{z\theta} \end{bmatrix} + \Delta T \begin{bmatrix} \alpha_r \\ \alpha_z \\ \alpha_\theta \\ \alpha_{rz} \\ 0 \\ 0 \end{bmatrix} \quad (1.21)$$

1.11 FORCE-DEFORMATION RELATIONSHIPS

The stress-strain equations presented in the previous sections are the fundamental *constitutive laws* for linear materials. However, for one-dimensional elements in structural engineering, we often rewrite these equations in terms of forces and deformations. For example, for a one-dimensional axially loaded member of length L and area A, the total axial deformation Δ and axial force P are $\Delta = L\varepsilon$ and $P = A\sigma$. Because $\sigma = E\varepsilon$, the force deformation relationship is:

$$P = k_a \Delta \tag{1.22}$$

where $k_a = \dfrac{AE}{L}$ and is defined as the axial stiffness of the member. Also, Equation (1.22) can be written in the following form:

$$\Delta = f_a P \tag{1.23}$$

where $f_a = \dfrac{L}{AE}$ and is defined as the axial flexibility of the member. It is important to note that the stiffness and flexibility terms are not a function of the load and are only the material and geometric properties of the member.

For a one-dimensional member of constant cross-section, the torsional force T in terms of the relative rotation φ between the ends of the member is given by:

$$T = k_T \varphi \tag{1.24}$$

where $k_T = \dfrac{JG}{L}$ in which J is the torsional moment of inertia. Also, the inverse of the torsional stiffness is the torsional flexibility.

In the case of pure bending of a beam fixed at one end, integration of a stress distribution over the cross-section produces a moment M. The linear strain distribution results in a rotation at the end of the beam of ϕ. For this finite length beam, the moment-rotation relationship is:

$$M = k_b \phi \tag{1.25}$$

where the bending stiffness $k_b = \frac{EI}{L}$. For a typical cross-section of the beam of length dx, the moment curvature relationship at location x is:

$$M(x) = EI\psi(x) \tag{1.26}$$

These force-deformation relationships are considered fundamental in the traditional fields of structural analysis and design.

1.12 SUMMARY

Material properties must be determined experimentally. Careful examinations of the properties of most structural materials indicate that they are not isotropic or homogeneous. Nonetheless, it is common practice to use the isotropic approximation for most analyses. In the future of structural engineering, however, the use of composite, anisotropic materials will increase significantly. The responsibility of the engineer is to evaluate the errors associated with these approximations by conducting several analyses using different material properties.

Remember the result obtained from a computer model is an estimation of the behavior of the real structure. The behavior of the structure is dictated by the fundamental laws of physics and is not required to satisfy the building code or the computer program's user manual.

1.13 REFERENCES

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