APPENDIX F

A DISPLACEMENT-BASED BEAM ELEMENT WITH SHEAR DEFORMATIONS

Never use a Cubic Function Approximation for a Non-Prismatic Beam

F.1 INTRODUCTION

{ XE "Shearing Deformations" }In this appendix a unique development of a displacement-based beam element with transverse shearing deformations is presented. The purpose of this formulation is to develop constraint equations that can be used in the development of a plate bending element with shearing deformations. The equations developed, which are based on a cubic displacement, apply to a beam with constant cross-section subjected to end loading only. For this problem both the force and displacement methods yield identical results.

To include shearing deformation in plate bending elements, it is necessary to constrain the shearing deformations to be constant along each edge of the element. A simple approach to explain this fundamental assumption is to consider a typical edge of a plate element as a deep beam, as shown in Figure F.1.

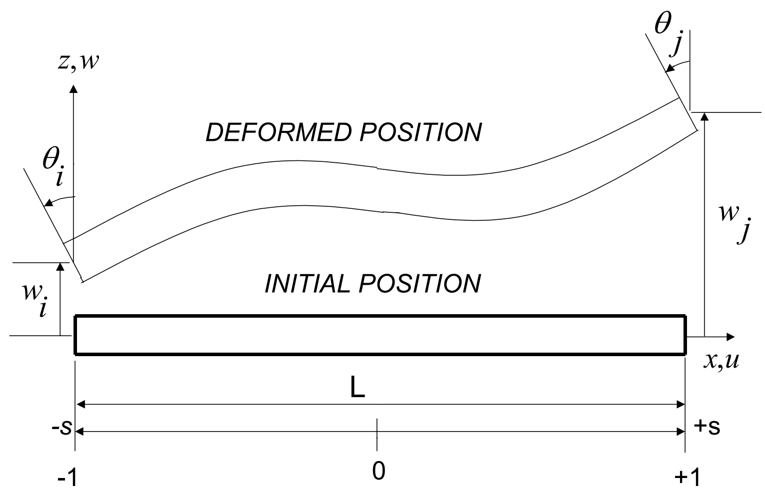

Figure F.1 Typical Beam Element with Shear Deformations

F.2 BASIC ASSUMPTIONS

In reference to Figure F.1, the following assumptions on the displacement fields are made:

First, the horizontal displacement caused by bending can be expressed in terms of the average rotation, θ, of the section of the beam using the following equation:

$$u = -z\,\theta \tag{F.1}$$

where z is the distance from the neutral axis.

Second, the consistent assumption for cubic normal displacement is that the average rotation of the section is given by:

$$\theta = N_1\theta_i + N_2\theta_j + N_3\Delta\theta \tag{F.2}$$

The cubic equation for the vertical displacement w is given by:

$$w = N_1 w_i + N_2 w_j + N_3 \beta_1 + N_4 \beta_2 \tag{F.3a}$$

where:

$$N_1 = \frac{1 - s}{2}, \quad N_2 = \frac{1 + s}{2}, \quad N_3 = 1 - s^2 \quad \text{and} \quad N_4 = s(1 - s^2) \tag{F.3b}$$

Note that the term $(1 - s^2)\Delta\theta$ is the relative rotation with respect to a linear function; therefore, it is a hierarchical rotation with respect to the displacement at the center of the element. One notes the simple form of the equations when the natural coordinate system is used.

It is apparent that the global variable x is related to the natural coordinate s by the equation $x = \frac{L}{2}s$. Therefore:

$$\partial x = \frac{L}{2}\partial s \tag{F.4}$$

Third, the elasticity definition of the "effective" shear strain is:

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}; \quad \text{hence,} \quad \gamma_{xz} = \frac{\partial w}{\partial x} - \theta \tag{F.5}$$

Because $\frac{\partial w}{\partial x} = \frac{2}{L}\frac{\partial w}{\partial s}$, the evaluation of the shear strain, Equation (F.5), produces an expression in terms of constants, a linear equation in terms of s and a parabolic equation in terms of s^2. Or:

$$\begin{aligned} \gamma_{xz} = {} & \frac{1}{L}(w_j - w_i) - \frac{4}{L}s\,\beta_1 - \frac{2}{L}(1 - 3s^2)\,\beta_2 \\ & - \frac{1 - s}{2}\theta_i - \frac{1 + s}{2}\theta_j - (1 - s^2)\alpha \end{aligned} \tag{F.6}$$

If the linear and parabolic expressions are equated to zero, the following constraint equations are determined:

$$\beta_1 = \frac{L}{8}(\theta_i - \theta_j) \tag{F.7a}$$

$$\beta_2 = \frac{L}{6}\Delta\theta \tag{F.7b}$$

The normal displacements, Equation (F.2), can now be written as:

$$w = N_1 w_i + N_2 w_j + N_3 \frac{L}{8}(\theta_i - \theta_j) + N_4 \frac{L}{6}\alpha \tag{F.8}$$

Also, the effective shear strain is constant along the length of the beam and is given by:

$$\gamma_{xz} = \frac{1}{L}(w_j - w_i) - \frac{1}{2}(\theta_i + \theta_j) - \frac{2}{3}\Delta\theta \tag{F.9}$$

Now, the normal bending strains for a beam element can be calculated directly from Equation (2.1) from the following equation:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{2z}{L}\frac{\partial \theta}{\partial s} = \frac{z}{L}[\theta_i - \theta_j + 4s\Delta\theta] \tag{F.10}$$

In addition, the bending strain ε_x can be written in terms of the beam curvature term ψ, which is associated with the section moment M. Or:

$$\varepsilon_x = z\psi \tag{F.11}$$

The deformation-displacement relationship for the bending element, including shear deformations, can be written in the following matrix form:

$$\begin{bmatrix} \psi \\ \gamma_{xz} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 4\,s \\ -L/\,2 & -L/\,2 & -1 & 1 & -2L/\,3 \end{bmatrix}\begin{bmatrix} \theta_i \\ \theta_j \\ w_i \\ w_j \\ \Delta\theta \end{bmatrix} \quad \text{or,}\ \ \mathbf{d} = \mathbf{B}\,\mathbf{u} \qquad \text{(F.12)}$$

The force-deformation relationship for a bending element is given by:

$$\begin{bmatrix} M \\ V \end{bmatrix} = \begin{bmatrix} \int z^2 E\, dA & 0 \\ 0 & \int \alpha\, G\, dA \end{bmatrix}\begin{bmatrix} \psi \\ \gamma_{xz} \end{bmatrix} \quad \text{or,}\ \ \sigma\mathbf{f} = \mathbf{C}\,\mathbf{d}\Delta \qquad \text{(F.13)}$$

where E is Young' s modulus, $\alpha\ G$ is the effective shear modulus and V is the total shear acting on the section.

The application of the theory of minimum potential energy produces a 5 by 5 element stiffness matrix of the following form:

$$\mathbf{K} = \frac{L}{2}\int \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\ ds \qquad \text{(F.14)}$$

Static condensation is used to eliminate $\Delta\theta$ to produce the 4 by 4 element stiffness matrix.

F.3 EFFECTIVE SHEAR AREA

{ XE "Effective Shear Area" }For a homogeneous rectangular beam of width "b" and depth "d," the shear distribution over the cross section from elementary strength of materials is given by:

$$\tau = [1 - \left(\frac{2z}{d}\right)^2]\ \tau_0 \qquad \text{(F.15)}$$

where, τ_0 is the maximum shear stress at the neutral axis of the beam. The integration of the shear stress over the cross section results in the following equilibrium equation:

$$\tau_0 = \frac{3}{bd} V \tag{F.16}$$

The shear strain is given by:

$$\gamma = \frac{1}{G}[1 - \left(\frac{2z}{d}\right)^2]\ \tau_0 \tag{F.17}$$

The internal strain energy per unit length of the beam is:

$$E_I = \frac{1}{2}\int \gamma\ \tau\ dA \quad = \quad \frac{3}{5bdG} V^2 \tag{F.18}$$

The external work per unit length of beam is:

$$E_E = \frac{1}{2} V\ \gamma_{xz} \tag{F.19}$$

Equating external to internal energy we obtain:

$$V = \frac{5}{6} G bd \gamma_{xz} \tag{F.20}$$

Therefore, the area reduction factor for a rectangular beam is:

$$\alpha = \frac{5}{6} \tag{F.21}$$

For non-homogeneous beams and plates, the same general method can be used to calculate the shear area factor.