11.

GEOMETRIC STIFFNESS AND P-DELTA EFFECTS

P-Delta Effects, Due To Dead Load, Can Be Considered Without Iteration for Both Static and Dynamic Analysis

11.1 DEFINITION OF GEOMETRIC STIFFNESS

{ XE "Geometric Stiffness" }{ XE "P-Delta Effects" }We are all aware that a cable has an increased lateral stiffness when subjected to a large tension force. If a long rod is subjected to a large compressive force and is on the verge of buckling, we know that the lateral stiffness of the rod has been reduced significantly and a small lateral load may cause the rod to buckle. This general type of behavior is caused by a change in the "geometric stiffness" of the structure. It is apparent that this stiffness is a function of the load in the structural member and can be either positive or negative.

{ XE "Cable Element" }The fundamental equations for the geometric stiffness for a rod or a cable are very simple to derive. Consider the horizontal cable shown in Figure 11.1 of length L with an initial tension T. If the cable is subjected to lateral displacements, v_i and v_j, at both ends, as shown, then additional forces, F_i and F_i , must be developed for the cable element to be in equilibrium in its displaced position. Note that we have assumed all forces and displacements are positive in the up direction. We have also made the assumption that the displacements are small and do not change the tension in the cable.

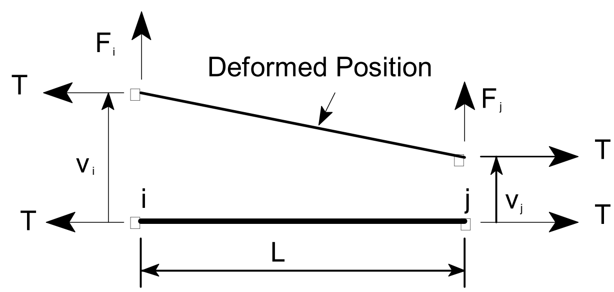


Figure 11.1 Forces Acting on a Cable Element

Taking moments about point j in the deformed position, the following equilibrium equation can be written:

$$F_i = \frac{T}{L}(v_i - v_j) \tag{11.1}$$

And from vertical equilibrium the following equation is apparent:

$$F_j = -F_i \tag{11.2}$$

Combining Equations 11.1 and 11.2, the lateral forces can be expressed in terms of the lateral displacements by the following matrix equation:

$$\begin{bmatrix} F_i \\ F_j \end{bmatrix} = \frac{T}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_i \\ v_j \end{bmatrix} \quad \text{or symbolically,} \quad \mathbf{F_g} = \mathbf{k_g v} \tag{11.3}$$

{ XE "Frame Element:Geometric Stiffness" }Note that the 2 by 2 geometric stiffness matrix, $\mathbf{k_g}$, is not a function of the mechanical properties of the cable and is only a function of the element's length and the force in the element. Hence, the term "geometric" or "stress" stiffness matrix is introduced so that the matrix has a different name from the "mechanical" stiffness matrix, which is based on the physical properties of the element. The geometric stiffness exists in all structures; however, it becomes important only if it is large compared to the mechanical stiffness of the structural system.

In the case of a beam element with bending properties in which the deformed shape is assumed to be a cubic function caused by the rotations ϕ_i and ϕ_j at the ends, additional moments M_i and M_j are developed. From Reference [1] the force-displacement relationship is given by the following equation:

$$\begin{bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{bmatrix} = \frac{T}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \begin{bmatrix} v_i \\ \phi_i \\ v_j \\ \phi_j \end{bmatrix} \quad \text{or,} \quad \mathbf{F}_G = \mathbf{k}_G \mathbf{v} \tag{11.4}$$

The well-known elastic force deformation relationship for a prismatic beam without shearing deformations is:

$$\begin{bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & -2L^2 \\ -12 & -6L & 12 & -6L \\ -6L & -2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_i \\ \phi_i \\ v_j \\ \phi_j \end{bmatrix} \quad \text{or,} \quad \mathbf{F}_E = \mathbf{k}_E \mathbf{v} \tag{11.5}$$

Therefore, the total forces acting on the beam element will be:

$$\mathbf{F}_T = \mathbf{F}_E + \mathbf{F}_G = [\mathbf{k}_E + \mathbf{k}_G]\mathbf{v} = \mathbf{k}_T \mathbf{v} \tag{11.6}$$

Hence, if the large axial force in the member remains constant, it is only necessary to form the total stiffness matrix, $\mathbf{k}_T$, to account for this stress stiffening or softening effect.

11.2 APPROXIMATE BUCKLING ANALYSIS

{ XE "Buckling Analysis" }In the case when the axial compressive force is large, $T = -P$, the total stiffness matrix of the beam can become singular. To illustrate this instability, consider the beam shown in Figure 11.2 with the displacements at point j set to zero.

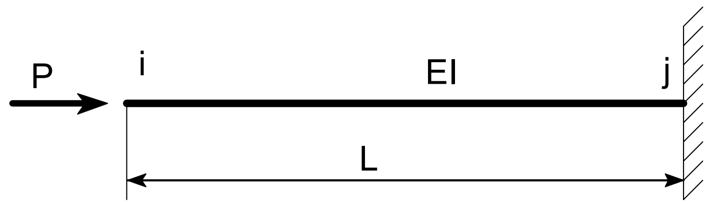


Figure 11.2 Cantilever Beam Subjected to Buckling Load

From Equation (11.6) the equilibrium equations for the beam shown in Figure 11.2 are in matrix form:

$$\begin{bmatrix} 12+36\lambda & 6L+3L\lambda \\ 6L+3L\lambda & 4L^2+4L^2\lambda \end{bmatrix} \begin{bmatrix} v_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11.7}$$

Where $\lambda = -\frac{PL^2}{30EI}$. This eigenvalue problem can be solved for the lowest root, which is:

$$\lambda_1 = -0.0858 \qquad \text{or} \qquad P_{cr} = 2.57\frac{EI}{L^2} \tag{11.8}$$

The well-known exact Euler buckling load for the cantilever beam is given by:

$$P_{cr} = \frac{\pi^2 EI}{4L^2} = 2.47\frac{EI}{L^2} \tag{11.9}$$

Therefore, the approximate solution Equation (11.8), which is based on a cubic shape, is within five percent of the exact solution.

If the straight line approximation is used, given by Equation (11.3), an approximate buckling load of $3.0\frac{EI}{L^2}$ is obtained. This is still a reasonable approximation.

11.3 P-DELTA ANALYSIS OF BUILDINGS

The use of the geometric stiffness matrix is a general approach to include secondary effects in the static and dynamic analysis of all types of structural systems. However, in Civil Structural Engineering it is commonly referred to as P-Delta Analysis that is based on a more physical approach. For example, in building analysis, the lateral movement of a story mass to a deformed position generates second-order overturning moments. This second-order behavior has been termed the P-Delta effect because the additional overturning moments on the building are equal to the sum of story weights "P" times the lateral displacements "Delta."

Many techniques have been proposed for evaluating this second-order behavior. Rutenberg [2] summarized the publications on this topic and presents a simplified method to include those second-order effects. Some methods consider the problem as one of geometric non-linearity and propose iterative solution techniques that can be numerically inefficient. Also, those iterative methods are not appropriate for dynamic analysis where the P-Delta effect causes lengthening of the periods of vibration. The equations presented in this section are not new. However, the simple approach used in their derivation should add physical insight to the understanding of P-Delta behavior in buildings [3].

The P-Delta problem can be linearized and the solution to the problem obtained ***directly*** and ***exactly***, without iteration, in building type structures where the weight of the structure is constant during lateral motions and the overall structural displacements can be assumed to be small compared to the structural dimensions. Furthermore, the additional numerical effort required is negligible.

The method does not require iteration because the total axial force at a story level is equal to the weight of the building above that level and does not change during the application of lateral loads. Therefore, the sum of the column of geometric stiffness terms associated with the lateral loads is zero, and only the axial forces caused by the weight of the structure need to be included in the evaluation of the geometric stiffness terms for the complete building.

The effects of P-Delta are implemented in the basic analytical formulation thus causing the effects to be consistently included in both static and dynamic analyses. The resulting structural displacements, mode shapes and frequencies include the effect of structural softening automatically. Member forces satisfy both static and dynamic equilibrium and reflect the additional P-Delta moments consistent with the calculated displacements.

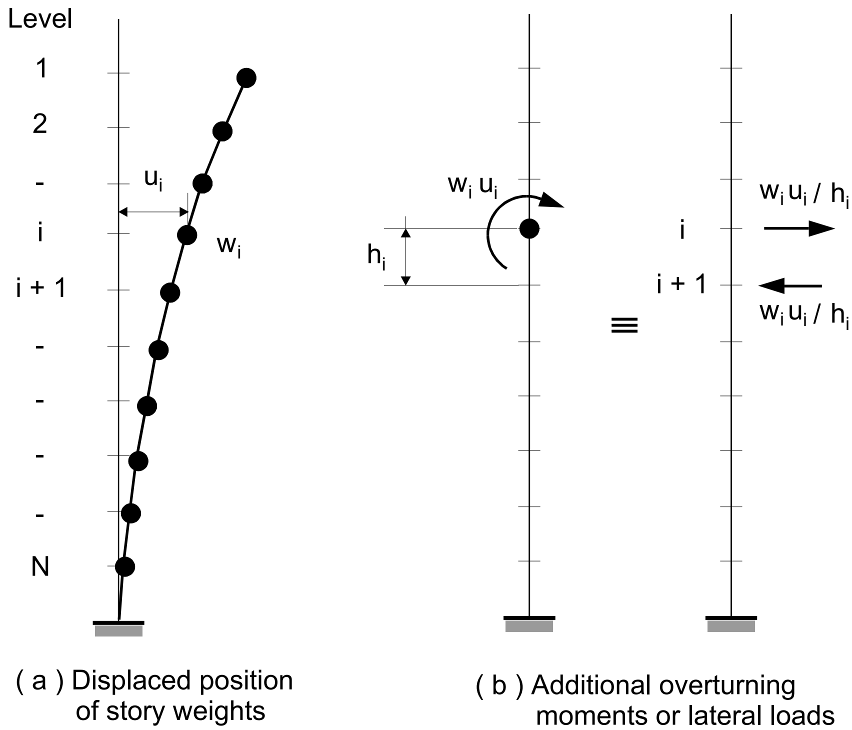


Figure 11.3 Overturning Loads Due to Translation of Story Weights

The vertical "cantilever type" structure shown in Figure 11.3 (a) is considered to illustrate the basic problem. Under lateral displacements, let us consider the additional overturning moments related to one mass, or story weight, at level i. The total overturning effects will be the sum of all story weight contributions. Figure 11.3 (b) indicates statically equivalent force systems that produce the same overturning moments. Or, in terms of matrix notation:

$$\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix} = \frac{w_i}{h_i} \begin{bmatrix} 1.0 \\ -1.0 \end{bmatrix} \begin{bmatrix} u_i \end{bmatrix} \tag{11.10}$$

The lateral forces shown in Figure 11.3 (b) can be evaluated for all stories and added to the external loads on the structure. The resulting lateral equilibrium equation of the structure is:

$$\mathbf{Ku} = \mathbf{F} + \mathbf{Lu} \tag{11.11}$$

where $\mathbf{K}$ is the lateral stiffness matrix with respect to the lateral story displacements $\mathbf{u}$. The vector $\mathbf{F}$ represents the known lateral loads and $\mathbf{L}$ is a matrix that contains w_i/h_i factors. Equation (11.11) can be rewritten in the form:

$$\mathbf{K}^{*}\mathbf{u} = \mathbf{F} \tag{11.12}$$

where $\mathbf{K}^{*} = \mathbf{K} - \mathbf{L}$

Equation (11.12) can be solved directly for the lateral displacements. If internal member forces are evaluated from these displacements, consistent with the linear theory used, it will be found that equilibrium with respect to the deformed position has been obtained. One minor problem exists with the solution of Equation (11.12); the matrix $\mathbf{K}^{*}$ is not symmetric. However, it can be made symmetric by replacing the lateral loads shown in Figure 11.3 (b) with another statically equivalent load system.

From simple statics the total contribution to overturning associated with the relative story displacement "u_i - u_{i+1}," can be written as:

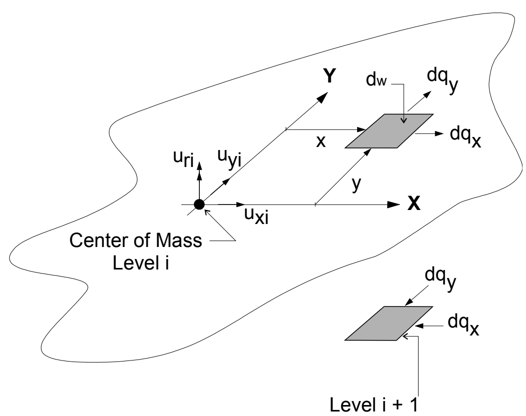
$$\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix} = \frac{W_i}{h_i} \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} \tag{11.13}$$

where W_i is the total dead load weight above story i. The $\mathbf{L}$ matrix is now symmetrical and no special non-symmetric equation solver is required.

It is of significant interest to note that Equation (11.13) is the exact form of the "geometric stiffness," Equation (11.3), for a column, including axial force effects only. Therefore, the physical development given here is completely equivalent to the more theoretical approach normally used to formulate the incremental stiffness in nonlinear structural analysis.

The equilibrium of a complete building can be formulated in terms of the lateral displacement of the floor level. Then, one can evaluate the contribution to the total geometric stiffness for each column at a particular story level in which the effects of the external lateral loads **F** are included in the evaluation of the axial forces in all columns. If this approach is used, the total geometric stiffness at the lateral equilibrium level is identical to Equation (11.13) because the lateral axial forces **F** do not produce a net increase in the total of all axial forces that exist in the columns at any level. Such a refined analysis must be iterative in nature; however, it does not produce more exact results.

It is clear that the beam-column stiffness effects as defined by Equation (11.4) have been neglected. The errors associated with those cubic shape effects can be estimated at the time member forces are calculated. However, the method presented here does include the overall large displacement side-sway behavior of



the complete structure that is associated with the global stability of the building.

Figure 11.4 Mass Distribution at Typical Floor Level

11.4 EQUATIONS FOR THREE-DIMENSIONAL BUILDINGS

Equation (11.13) can be applied directly in both directions for buildings in which the centroids are the same for all story levels. However, for the more general building, the equations for the story couples are more complicated. A general three-dimensional building system is shown schematically in Figure 11.4.

It is assumed that the three-dimensional building stiffness of the system has been formulated with respect to the two lateral displacements, u_{xi}, u_{yi}, and rotation, u_{ri}, at the center of mass at each story level. In addition to the overturning forces given by Equation (11.13), secondary forces exist because of the distribution of the story mass over a finite floor size.

The first step before developing the 6 by 6 geometric stiffness matrix for each story is to calculate the location of the center of mass and the rotational moment of inertia for all story levels. For a typical story i, it is then necessary to calculate the total weight and centroid of the structure above that level. Because of the relative displacements between story i and story i + 1, from Equation 11.13, forces must be developed to maintain equilibrium. Those forces and displacements must then be transformed to the center of mass at both level i and i + 1.

11.5 THE MAGNITUDE OF P-DELTA EFFECTS

The comparison of the results of two analyses with and without P-Delta will illustrate the magnitude of the P-Delta effects. A well-designed building usually has well-conditioned level-by-level stiffness/weight ratios. For such structures, P-Delta effects are usually not very significant. The changes in displacements and member forces are less than 10%.

However, if the weight of the structure is high in proportion to the lateral stiffness of the structure, the contributions from the P-Delta effects are highly amplified and, under certain circumstances, can change the displacements and member forces by 25 percent or more. Excessive P-Delta effects will eventually introduce singularities into the solution, indicating physical structure instability. Such behavior is clearly indicative of a poorly designed structure that is in need of additional stiffness.

An analysis of a 41-story steel building was conducted with and without P-Delta effects. The basic construction was braced frame and welded steel shear wall.

The building was constructed in a region where the principal lateral loading is wind. The results are summarized in Table 11.1.

Table 11.1 P-Delta Effects on Typical Building

	Without P-Delta	With P-Delta
First Mode Period (seconds)	5.33	5.52
Second Mode Period (seconds)	4.21	4.30
Third Mode Period (seconds)	4.01	4.10
Fourth Mode Period (seconds)	1.71	1.75
Wind Displacement (inches)	7.99	8.33

Because the building is relatively stiff, the P-Delta effects are minimal. Also, it is apparent that P-Delta effects are less important for higher frequencies.

11.6 P-DELTA ANALYSIS WITHOUT COMPUTER PROGRAM MODIFICATION

Many engineers are using general purpose, structural analysis programs for buildings that cannot be easily modified to include the equations presented here. Equation 11.4 presents the form of the lateral force-displacement equations for story i. We note that the form of this 2 x 2 geometric stiffness matrix is the same as the stiffness matrix for a prismatic column that has zero rotations at the top and bottom. Therefore, it is possible to add "dummy columns" between story levels of the building and assign appropriate properties to achieve the same effects as the use of geometric stiffness [2]. The force-displacement equations of the "dummy column" are:

$$\begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix} = \frac{12EI}{h_i^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} \qquad (11.14)$$

Therefore, if the moment of inertia of the column is selected as:

$$I = -\frac{W_i h_i^2}{12E} \tag{11.15}$$

The dummy column will have the same negative stiffness values as the linear geometric stiffness.

11.7 EFFECTIVE LENGTH - K FACTORS

{ XE "Effective Length" }{ XE "K Factor" }The solution procedure for the P-Delta effects described in this chapter has been implemented and verified in the ETABS program. The application of the method of analysis presented in this chapter should lead to the elimination of the column effective length (K-) factors, since the P-Delta effects automatically produce the required design moment amplifications. Also, the K-factors are approximate, complicated, and time-consuming to calculate. Building codes for concrete [4] and steel [5] now allow explicit accounting of P-Delta effects as an alternative to the more involved and approximate methods of calculating moment magnification factors for most column designs.

11.8 GENERAL FORMULATION OF GEOMETRY STIFFNESS

{ XE "Geometric Stiffness" }{ XE "Strain Displacement Equations:3D Nonlinear Solids" }It is relatively simple to develop the geometric stiffness matrix for any type of displacement-based finite element [1]. It is only necessary to add to the linear strain-displacement equations, Equations (2.3a-f), the higher order nonlinear terms. These large strain equations, in a local x-y-z reference system, are:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\bar{\mathbf{u}}_{,x}^T \bar{\mathbf{u}}_{,x} \\
\varepsilon_y &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\bar{\mathbf{u}}_{,y}^T \bar{\mathbf{u}}_{,y} \\
\varepsilon_z &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\bar{\mathbf{u}}_{,z}^T \bar{\mathbf{u}}_{,z} \\
\gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{1}{2}\bar{\mathbf{u}}_{,x}^T \bar{\mathbf{u}}_{,y} + \frac{1}{2}\bar{\mathbf{u}}_{,y}^T \bar{\mathbf{u}}_{,x} \\
\gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} + \frac{1}{2}\bar{\mathbf{u}}_{,x}^T \bar{\mathbf{u}}_{,z} + \frac{1}{2}\bar{\mathbf{u}}_{,z}^T \bar{\mathbf{u}}_{,x} \\
\gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x} + \frac{1}{2}\bar{\mathbf{u}}_{,y}^T \bar{\mathbf{u}}_{,z} + \frac{1}{2}\bar{\mathbf{u}}_{,z}^T \bar{\mathbf{u}}_{,y}
\end{aligned}
\tag{11.16}
$$

The nonlinear terms are the product of matrices that are defined as:

$$
\bar{\mathbf{u}}_{,x} = \begin{bmatrix} u_{x,x} \\ u_{y,x} \\ u_{z,x} \end{bmatrix}, \quad \bar{\mathbf{u}}_{,y} = \begin{bmatrix} u_{x,y} \\ u_{y,y} \\ u_{z,y} \end{bmatrix}, \quad \bar{\mathbf{u}}_{,z} = \begin{bmatrix} u_{x,z} \\ u_{y,z} \\ u_{z,z} \end{bmatrix} \tag{11.17}
$$

Equation (11.16) can be expressed in terms of the following sum of linear and nonlinear components:

$$
\mathbf{d} = \mathbf{d}_L + \mathbf{d}_N \tag{11.18}
$$

These strain-displacement equations, written in terms of engineering strains and in matrix notation, are identical to the classical Green-Lagrange strains. This is often referred to as the total Lagrangian approach in which the strains are computed with respect to the original reference system and the large rigid-body rotation is exact.

Using the same shape functions as used to form the element stiffness matrix, the derivatives of the displacements can be written as:

$$
\mathbf{g} = \mathbf{G}\mathbf{u} \tag{11.19}
$$

If the initial stresses are large, the potential energy of the structure must be modified by the addition of the following term:

$$\Omega_\sigma = \frac{1}{2}\int \begin{bmatrix} \bar{\mathbf{u}}_{,x}^T & \bar{\mathbf{u}}_{,y}^T & \bar{\mathbf{u}}_{,z}^T \end{bmatrix} \begin{bmatrix} \mathbf{s}_{xx} & \mathbf{s}_{xy} & \mathbf{s}_{xz} \\ \mathbf{s}_{yx} & \mathbf{s}_{yy} & \mathbf{s}_{yz} \\ \mathbf{s}_{zx} & \mathbf{s}_{zy} & \mathbf{s}_{zz} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_{,x} \\ \bar{\mathbf{u}}_{,y} \\ \bar{\mathbf{u}}_{,z} \end{bmatrix} dV = \frac{1}{2}\int \mathbf{g}^T \mathbf{S}\,\mathbf{g}\, dV \quad (11.20)$$

The 3 by 3 initial stress matrices are of the following form:

$$\mathbf{s}_{ij} = \begin{bmatrix} \sigma_{ij} & 0 & 0 \\ & \sigma_{ij} & 0 \\ 0 & 0 & \sigma_{ij} \end{bmatrix}_0 \quad (11.21)$$

where the initial stresses are defined as:

$$\mathbf{s}_0^T = \lfloor \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{xz} \quad \sigma_{yz} \rfloor_0 \quad (11.22)$$

Therefore, the geometric stiffness for any element can be calculated from:

$$\mathbf{k}_g = \int \mathbf{G}^T \mathbf{S}\,\mathbf{G}\, dV \quad (11.23)$$

For most finite elements the geometric stiffness is evaluated by numerical integration.

11.9 SUMMARY

The SAP2000 program has the option to add a three-dimensional geometric stiffness matrix to each frame element. Therefore, guyed towers, cable stay and suspension bridges can be modeled if the tension in the cable is not modified by the application of the load. If the initial axial forces in the elements are significantly changed by the addition of loads, iteration may be required. However, in the case of dynamic analysis, the evaluation of the eigen or LDR vectors must be based on one set of axial forces.

Most traditional methods for incorporating P-Delta effects in analysis of buildings are based on iterative techniques. Those techniques are time-consuming and are, in general, used for static analysis only. For building structures, the mass that causes the P-Delta effect is constant irrespective of the lateral loads and displacements. This information is used to linearize the P-Delta effect for buildings and solve the problem "exactly," satisfying equilibrium in the deformed position without iterations. An algorithm is developed that incorporates P-Delta effects into the basic formulation of the structural stiffness matrix as a geometric stiffness correction. This procedure can be used for static and dynamic analysis and will account for the lengthening of the periods and changes in mode shapes caused by P-Delta effects.

A well designed building should not have significant P-Delta effects. Analyses ***with and without the P-Delta effects*** will yield the magnitude of the P-Delta effects separately. If those lateral displacements differ by more than 5% ***for the same lateral load***, the basic design may be too flexible and a redesign should be considered.

The current SEAOC Blue Book states "the drift ratio of $0.02/R_W$ serves to define the threshold of deformation beyond which there may be significant P-Delta effects." Clearly, if one includes P-Delta effects in all analyses, one can disregard this statement. If the loads acting on the structure have been reduced by a ductility factor R_W, however, the P-Delta effects should be amplified by R_W to reflect ultimate load behavior. This can be automatically included in a computer program using a multiplication factor for the geometric stiffness terms.

It is possible to calculate geometric stiffness matrices for all types of finite elements. The same shape functions used in developing the elastic stiffness matrices are used in calculating the geometric stiffness matrix.

11.10 REFERENCES

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