9.

MEMBRANE ELEMENT WITH NORMAL ROTATIONS

Rotations Must Be Compatible Between Beam, Membrane and Shell Elements

9.1 INTRODUCTION

{ XE "Membrane Element" }The complex nature of most buildings and other civil engineering structures requires that frame, plate bending and membrane elements exist in the same computer model. The three-dimensional beam element normally has six degrees-of-freedom per node—three displacements and three rotations per node. The plate bending element, presented in the previous chapter, has two rotations in the plane of the element and one displacement normal to the element at each node. The standard plane stress element, used to model the membrane behavior in shell elements, has only two in-plane displacements at each node and cannot carry moments applied normal to the plane of the element.

{ XE "Normal Rotations" }A frame element embedded normal to a shear wall or slab is very common in the modeling of buildings and many other types of structural systems. It is possible to use a constraint to transfer the frame element moment to a force-couple applied in the plane of the element. However, for shells connected to edge beams and many other common types of structural systems, there is a need for a membrane element that has a normal rotation as a basic DOF at each node.

{ XE "Quadrilateral Element" }The search for a membrane element with normal rotations was a fruitless endeavor for the first 30 years of the development of finite element technology. Within the last 15 years, however, a practical quadrilateral element has evolved. Rather than refer to the many research papers (summarized in reference [1]) that led to the development of the element currently used in the general structural analysis program SAP2000, the fundamental equations will be developed in this chapter. In addition, numerical examples will be presented to illustrate the accuracy of the element.

9.2 BASIC ASSUMPTIONS

The development of the membrane element is very similar to the plate bending element presented in the previous chapter. The quadrilateral element is shown in Figure 9.1.

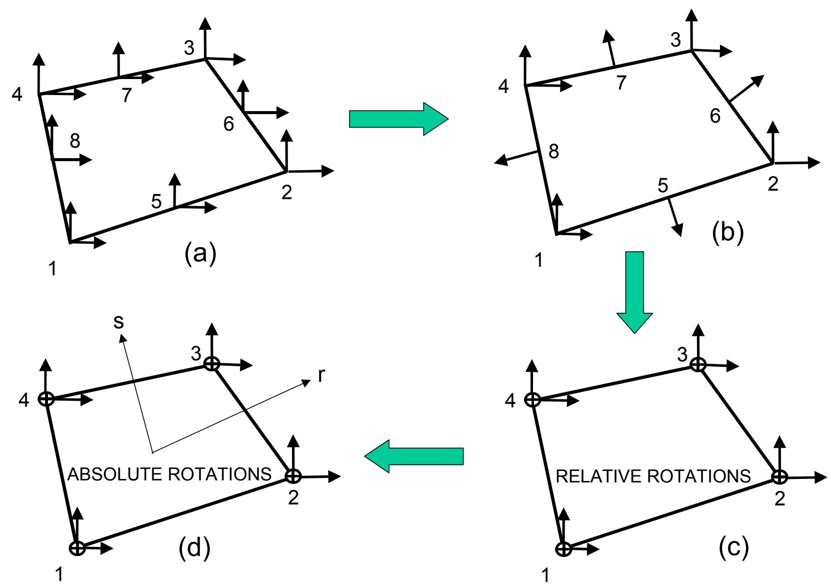


Figure 9.1 Quadrilateral Membrane Element with Normal Rotations

Development of the element can be divided into the following four steps:

1. The starting point is the nine-node quadrilateral element, 16 DOF, shown in Figure 9.1a.

2. The next step is to rotate the mid-side relative displacements to be normal and tangential to each side and to set the relative tangential displacement to zero, reducing the element to the 12 DOF shown in Figure 9.1b.
3. The third step is to introduce parabolic normal displacement constraints to eliminate the four mid-side normal displacements and to introduce four relative normal rotations at the nodes shown in Figure 9.1c.
4. The final step is to convert the relative normal rotations to absolute values and to modify the shape functions to pass the patch test. This results in the 12 by 12 element stiffness with respect to the 12 DOF shown in Figure 9.1d.

9.3 DISPLACEMENT APPROXIMATION

The basic assumption is that in-plane x and y displacements are defined by the following equations:

$$
\begin{aligned}
u_x(r,s) &= \sum_{i=1}^{4} N_i(r,s)u_{xi} + \sum_{i=5}^{8} N_i(r,s)\ \Delta u_{xi} \\
u_y(r,s) &= \sum_{i=1}^{4} N_i(r,s)u_{yi} + \sum_{i=5}^{8} N_i(r,s)\ \Delta u_{yi}
\end{aligned}
\tag{9.1}
$$

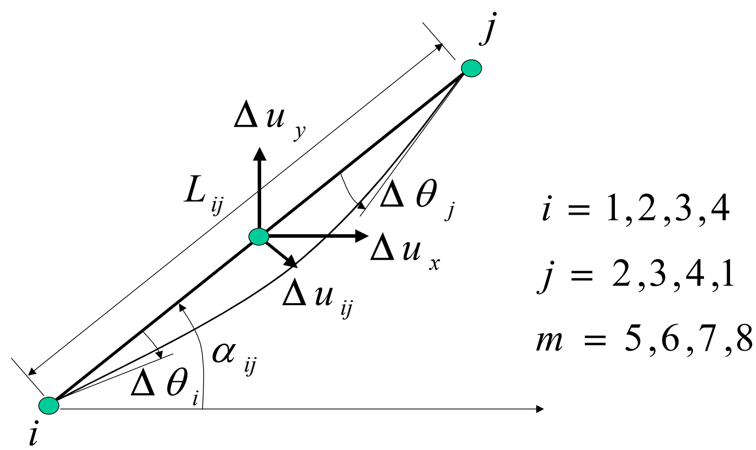
The eight shape functions are given by:

$$
\begin{aligned}
N_1 &= (1-r)(1-s)/\ 4 & N_2 &= (1+r)(1-s)/\ 4 \\
N_3 &= (1+r)(1+s)/\ 4 & N_4 &= (1-r)(1+s)/\ 4 \\
N_5 &= (1-r^2)(1-s)/\ 2 & N_6 &= (1+r)(1-s^2)/\ 2 \\
N_7 &= (1-r^2)(1+s)/\ 2 & N_8 &= (1-r)(1-s^2)/\ 2
\end{aligned}
\tag{9.2}
$$

The first four shape functions are the natural bilinear shape functions for a four-node quadrilateral and are not zero at nodes 5 to 8. The last four shape functions for the mid-side nodes and center node are an addition to the bilinear functions and are referred to as ***hierarchical*** functions.

9.4 INTRODUCTION OF NODE ROTATION

A typical element side *ij* is shown in Figure 9.2.



$$L_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

Figure 9.2 Typical Side of Quadrilateral Element

If it is assumed that the relative normal displacement of the side is parabolic, the following equation must be satisfied:

$$\Delta u_{ij} = \frac{L_{ij}}{8}(\Delta\theta_j - \Delta\theta_i) \tag{9.3}$$

Because the tangential mid-side displacement is zero, the global relative mid-side displacements are given by:

$$\begin{aligned}\Delta u_x &= \cos\alpha_{ij}\Delta u_{ij} = \cos\alpha_{ij}\frac{L_{ij}}{8}(\Delta\theta_j - \Delta\theta_i)\\ \Delta u_y &= -\sin\alpha_{ij}\Delta u_{ij} = -\sin\alpha_{ij}\frac{L_{ij}}{8}(\Delta\theta_j - \Delta\theta_i)\end{aligned} \tag{9.4}$$

Equation (9.4) can be applied to all four sides and the global displacements, Equation (9.1), can be written as:

$$\begin{aligned}u_x(r,s) &= \sum_{i=1}^{4} N_i(r,s)u_{xi} + \sum_{i=5}^{8} M_{xi}(r,s)\ \Delta\theta_i\\ u_y(r,s) &= \sum_{i=1}^{4} N_i(r,s)u_{yi} + \sum_{i=5}^{8} M_{yi}(r,s)\ \Delta\theta_i\end{aligned} \tag{9.5}$$

Therefore, the system has been reduced to 12 DOF.

9.5 STRAIN-DISPLACEMENT EQUATIONS

{ XE "Strain Displacement Equations:2D Plane Elements" }The strain-displacement equations can now be constructed from the following fundamental equations:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y} \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{9.6}$$

Alternatively, the 3 by 12 strain-displacement equations written in sub matrix form are the following:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \Delta\theta \end{bmatrix} \tag{9.7}$$

{ XE "Correction Matrix" }In order that the element satisfies the constant stress patch test, the following modification to the 3 by 4 $\mathbf{B}_{12}$ matrix must be made:

$$\overline{\mathbf{B}}_{12} = \mathbf{B}_{12} - \frac{1}{A}\int \mathbf{B}_{12}\, dA \tag{9.8}$$

The development of this equation is presented in the chapter on incompatible elements, Equation (6.4).

9.6 STRESS-STRAIN RELATIONSHIP

{ XE "Stress-Strain Relationship" }The stress-strain relationship for orthotropic plane stress materials can be written as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{9.9}$$

The only restriction on the stress-strain matrix is that it must be symmetric and positive definite.

9.7 TRANSFORM RELATIVE TO ABSOLUTE ROTATIONS

The element 12 by 12 stiffness matrix for a quadrilateral element with normal rotations is obtained using four-point numerical integration. Or:

$$\overline{\mathbf{K}} = \int \mathbf{B}^{\mathrm{T}} D \mathbf{B} \, dV \tag{9.10}$$

{ XE "Relative Rotations" }The stiffness matrix for the membrane element, as calculated from Equation (9.9), has four unknown relative rotations at the nodes. An examination of the properties of the stiffness matrix indicates that it has a zero energy mode in addition to the three rigid body modes. This spurious deformation mode, relative to the rigid-body rotation of the element, is shown in Figure 9.3.

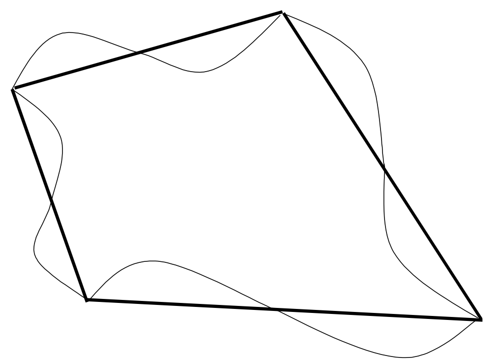


Figure 9.3 Zero Energy Displacement Mode

{ XE "Zero Energy Mode" }The zero energy displacement mode has equal rotations at all nodes and zero mid-side displacements. To eliminate this mode, it is only necessary to add a rank one matrix to the element stiffness matrix that has stiffness associated with the mode. From the elasticity definition of rotation, the absolute rotation at the center of the element, or an estimation of the rigid-body rotation of the element, can be calculated from:

$$\theta_0 = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right] = \mathbf{b}_0 \mathbf{u} \tag{9.11}$$

{ XE "Relative Rotations" }where $\mathbf{b}_0$ is a 1 by 12 matrix. The difference between the absolute rotation and the average relative rotation at the center of the element is:

$$d = \theta_0 - \sum_{i=1}^{4} N_i(0,0)\Delta\theta_i = \overline{\mathbf{b}}_0 \, \mathbf{u} \tag{9.12}$$

A stiffness k_0 (or a penalty term) can now be assigned to this deformation to create, using one point integration, the following rank one stiffness matrix:

$$\mathbf{K}_0 = \int \overline{\mathbf{b}}_0^T k_0 \, \overline{\mathbf{b}}_0 \, dV = k_0 \, Vol \, \overline{\mathbf{b}}_0^T \, \overline{\mathbf{b}}_0 \tag{9.13}$$

Experience with the solution of a large number of problems indicates that the following value for rotational stiffness is effective:

$$k_0 = 0.025 D_{33} \tag{9.14}$$

where D_{33} is the shear modulus for isotropic materials. When this rank one matrix is added to the 12 by 12 stiffness matrix, the zero energy mode is removed and the node rotation is converted to an absolute rotation.

9.8 TRIANGULAR MEMBRANE ELEMENT

{ XE "Triangular Elements" }The same approximations used to develop the quadrilateral element are applied to the triangular element with three mid-side nodes. The resulting stiffness matrix is 9 by 9. Approximately 90 percent of the computer program for the quadrilateral element is the same as for the triangular element. Only different shape functions are used and the constraint associated with the fourth side is skipped. However, the triangle is significantly more stiff than the quadrilateral. In fact, the accuracy of the membrane behavior of the triangle with the drilling degrees of freedom is nearly the same as the constant strain triangle.

9.9 NUMERICAL EXAMPLE

{ XE "Distorted Elements" }The beam shown in Figure 9.4 is modeled with two membrane elements with drilling degrees-of-freedom.

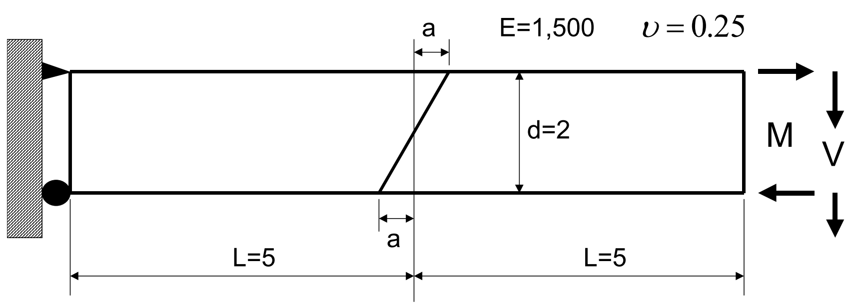


Figure 9.4 Beam Modeled with Distorted Elements

Results for both displacements and stresses are summarized in Table 9.1.

Table 9.1. Results of Analysis of Cantilever Beam

Mesh Distortion Factor "a"	TIP MOMENT LOADING		TIP SHEAR LOADING	
	Normalized Tip Displacement	Normalized Maximum Stress At Support	Normalized Tip Displacement	Normalized Maximum Stress At Support
Exact	1.000	1.000	1.000	1.000
0	1.000	1.000	0.958	0.750
1	0.502	0.675	0.510	0.601
2	0.280	0.627	0.303	0.557

{ XE "Shear Locking" }For rectangular elements subjected to end moment, the exact results are obtained and "shear locking" does not exist. For a tip shear loading, the displacements are in error by only 4 percent; however, the bending stresses are in error by 25 percent. This behavior is almost identical to the behavior of plane elements with incompatible modes. As the element is distorted, the displacements and stresses deteriorate. All results were obtained using four-point integration.

The end moment can be applied as two equal and opposite horizontal forces at the end of the beam. Or, one half of the end moment can be applied directly as two concentrated moments at the two end nodes. The results for the two different methods of loading are almost identical. Therefore, standard beam elements can be attached directly to the nodes of the membrane elements with normal rotational DOF.

9.10 SUMMARY

The membrane plane stress element presented in this chapter can be used to accurately model many complex structural systems where frame, membrane and plate elements interconnect. The quadrilateral element produces excellent results. However, the performance of the triangular membrane element is very poor.

9.11 REFERENCES

1. { XE "Ibrahimbegovic, Adnan" }Ibrahimbegovic, Adnan, R. Taylor, and E. Wilson. 1990. "A Robust Membrane Quadrilateral Element with Drilling Degrees of Freedom," *Int. J. of Num. Meth.*